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Reciprocity Inequalities, Coherence Time and Bandwidth in Signal Analysis and Optics

E. WOLF

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RECIPROCITY INEQUALITIES, COHERENCE TIME AND BANDWIDTH IN SIGNAL ANALYSIS AND OPTICS

E. Wolf

Emil Wolf

E. Wolf

Morris Kline

Morris Kline
Project Director

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Abstract

This paper is concerned with establishing a rigorous reciprocity relation of the Heisenberg type between the coherence time and the bandwidth of polychromatic radiation. Following upon an observation of Gabor[2] that the usual formulation of the reciprocity inequality which relates the effective duration of a signal and its effective bandwidth is unsatisfactory for signals that are intrinsically real (e.g., an electric signal) a modified version is here presented. Our analysis is closely related to that of Gabor and supplements it in several respects. In particular we show that his formulation, like our own, is restricted to signals with zero mean value. Contrary to general belief, there is no evidence that the inequality holds when this restriction is removed.

A definition of the coherence time of light is then proposed in terms of the (measurable) correlation function that plays a central part in the theory of partial coherence. The coherence time obeys rigorously a reciprocity inequality of the required type and is shown to have a simple interpretation in terms of averages involving the correlations in fluctuations of the instantaneous intensity. It is also shown that the coherence time may be determined experimentally from Michelson's visibility curve of interference fringes.

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1. Introduction

When two beams of light originating in the same source are allowed to interfere, the visibility of the interference fringes decreases (in general not monotonically) as the path difference between them is increased and eventually the fringes disappear altogether. It is well known (cf. Born[1], §42) that under usual experimental conditions, the time $\Delta t = \Delta \ell / c$ (c = velocity of light) that corresponds to the path difference $\Delta \ell$ at which the fringes disappear, and the effective spectral frequency range $\Delta \nu$ are connected by the order of magnitude relation

$$\Delta t \Delta \nu \sim 1. \tag{1.1}$$

Δt is called the coherence time and $\Delta \ell$ the coherence length of the light.

In analogy with the well-known mathematical inequality that is intimately connected with the quantum mechanical uncertainty relation of Heisenberg, one might suspect that the coherence time and the effective bandwidth could be defined more precisely so as to yield a rigorous reciprocity inequality of the form

$$\Delta t \Delta \nu \geq a, \tag{1.2}$$

where a is a constant of the order of unity.

When one tries to apply to the present problem an argument similar to that which leads to the usual inequality, two unsatisfactory features are revealed. In the first place, unlike the wave functions of quantum mechanics, the basic functions that characterize an optical field (components of the electric and magnetic field vectors) are intrinsically real; hence if such a function is represented by a Fourier integral, say

$$V(t) = \int_{-\infty}^{+\infty} v(\nu) e^{-2\pi i \nu t} d\nu, \tag{1.3}$$

then, because of the 'reality condition' $v(-\nu) = v^*(\nu)$ (asterisks denoting the complex conjugate), the mean frequency

$$\bar{\nu} = \frac{\int_{-\infty}^{+\infty} \nu \, v(\nu) v^*(\nu) d\nu}{\int_{-\infty}^{+\infty} v(\nu) v^*(\nu) d\nu} \tag{1.4}$$

turns out to be zero. In consequence the root-mean-square (r.m.s.) spectral width $\Delta\nu$, defined by the formula

$$(1.5) \qquad (\Delta\nu)^2 = \frac{\int_{-\infty}^{+\infty} (\nu-\bar{\nu})^2 v(\nu)v^*(\nu)d\nu}{\int_{-\infty}^{+\infty} v(\nu)v^*(\nu)d\nu}$$

is centered on a mean value ($\bar{\nu}$) that is quite different from what is normally understood by the mean frequency of a real signal; for in the domain of classical, non-relativistic physics, only positive frequencies are considered to have a physical significance and the mean frequency and the r.m.s. width ought to be defined by formulas of the type (1.4) and (1.5) but with the lower limit $-\infty$ replaced by zero. However, when this modification is made, the symmetry between the time representation and the frequency representation is destroyed and the usual argument leading to the reciprocity inequality no longer applies. This fact appears to have been first noted by Gabor[2] in his well-known paper on communication theory. Gabor suggested that an inequality of the type (1.2) lies at the root of a fundamental principle of communication theory and he showed that a more satisfactory formulation of the inequality is obtained if the averages $\bar{\nu}$ and $\Delta\nu$, and the corresponding averages $\bar{t}$ and Δt in the time domain, are formally defined in terms of the so-called complex analytic signal, which is a certain linear transform [see Section 2 below] containing only positive spectral frequencies of the real signal V(t). Unfortunately, Gabor's discussion does not reveal the physical significance of the quantities $\bar{t}$ and Δt introduced in this way.

The second unsatisfactory aspect, of significance for optics rather than for communication theory in general arises from the fact that the 'signal' V(t) is never measured in an optical experiment. Only certain averages over time intervals that are very large compared to the basic period $1/\bar{\nu}$ are measureable*. Hence an inequality involving quantities that are defined directly in terms of V(t) cannot, at optical frequencies at any rate, be tested experimentally.

In this paper a rigorous inequality of the required type free of these objections is established. This is achieved by defining Δt in terms of the auto-correlation function of the disturbance V and $\Delta\nu$ in terms of its Fourier transform, the power spectrum of V. The former may be determined from the measurements of the Michelson visibility curve, the latter from measurements of the spectral energy function. The 'coherence time' Δt defined in this way has a simple interpretation in terms of an average relating to the correlation between

*We may recall that the optical periods are of the order of 10^{-15} sec., whereas the best photo-multipliers that are at present available have resolving time of the order of 10^{-9} sec.

intensity fluctuations, rather than in terms of amplitude fluctuations.

As a preliminary, Gabor's proof is completed and it is shown that the quantity Δt defined by him (to be denoted $\Delta' t$ here) may be interpreted as the normalized r.m.s. of the squared amplitude of the envelope belonging to the real signal. Moreover, it will be shown that if $\Delta' t$ exists, it is equal to the normalized r.m.s. width Δt of the square of the real signal itself. However, the requirement of the existence of $\Delta' t$ is found to be a rather strong one. For example, if the real signal $V(t)$ is zero outside a finite interval $-T \leq t \leq T$, then Δt exists but $\Delta' t$ does not exist unless the average value of the signal is zero, or what amounts to the same thing, unless the signal has no spectral components of zero frequency. This condition is satisfied in the application discussed in this paper, but the need for imposing it raises doubts about the validity of the inequality (1.2) for an arbitrary real signal when $\bar{\nu}$ and $\Delta\nu$ are redefined in accordance with our intuitive notion of these quantities.*

2. The analytic signal and Gabor's formulation of the inequality for real signals

Let $V(t)$ $(-\infty < t < \infty)$ be a real signal, assumed to possess a Fourier integral representation

$$V(t) = \int_{-\infty}^{+\infty} v(\nu)\, e^{-2\pi i \nu t} d\nu. \tag{2.1}$$

Then, by the Fourier inversion formula

$$v(\nu) = \int_{-\infty}^{+\infty} V(t)\, e^{2\pi i \nu t} dt. \tag{2.2}$$

Since $V(t)$ is real,

$$v(-\nu) = v^*(\nu). \tag{2.3}$$

The complex analytic signal** $\hat{V}(t)$ associated with $V(t)$ is defined by the equation

---

*See paper by Kay and Silverman[3] referred to after equation (3.15b).

**Also called 'half-range complex function'.

$$\hat{V}(t) = 2\int_0^\infty v(\nu)\, e^{-2\pi i\nu t} d\nu, \tag{2.4}$$

i.e., $\hat{V}$ is obtained from V by suppressing all the negative frequency terms and multiplying the amplitudes of the positive terms by two.

Let $|v(\nu)|$ be the amplitude and $\phi(\nu)$ the phase of $v(\nu)$:

$$v(\nu) = |v(\nu)|\, e^{i\phi(\nu)}. \tag{2.5}$$

Using (2.3) it is easily seen that V and $\hat{V}$ may also be written in the form

$$V(t) = 2\int_0^\infty |v(\nu)| \cos\left[\phi(\nu) - 2\pi\nu t\right] d\nu, \tag{2.6}$$

$$\hat{V}(t) = 2\int_0^\infty |v(\nu)|\, e^{i\left[\phi(\nu)-2\pi\nu t\right]} d\nu. \tag{2.7}$$

Evidently

$$\hat{V}(t) = V(t) + iW(t), \tag{2.8}$$

where

$$W(t) = 2\int_0^\infty |v(\nu)| \sin\left[\phi(\nu)-2\pi\nu t\right] d\nu. \tag{2.9}$$

V(t) and W(t) are <u>allied Fourier integrals</u>, and it is well known[4] that they are Hilbert transforms of each other:

$$W(t) = \frac{1}{\pi} P\int_{-\infty}^{+\infty} \frac{V(t')}{t'-t}\, dt', \qquad V(t) = -\frac{1}{\pi} P\int_{-\infty}^{+\infty} \frac{W(t')}{t'-t}\, dt', \tag{2.10}$$

where P denotes the Cauchy principal value at $t' = t$. From (2.8) we see that the real part of the analytic signal associated with a real signal V is this signal itself, while the imaginary part is its Hilbert transform.

In terms of the improper function

$$\delta_-(x) = \int_0^\infty e^{2\pi i\nu x} d\nu\,, \tag{2.11}$$

well known in quantum mechanics (cf. Heitler[5], p. 69), the relation between V and $\hat{V}$ may be expressed in the form of a linear transform

$$\hat{V}(t) = 2\int_{-\infty}^{\infty} V(t')\delta_{-}(t'-t)dt', \tag{2.12}$$

which follows on substituting from (2.2) into (2.4).

In Gabor's formulation of the reciprocity inequality for real signals, the mean frequency $\bar{\nu}$ and the r.m.s. $\Delta\nu$ are defined not by the customary formulas (1.4) and (1.5) but by the more appropriate formulas*

$$\bar{\nu} = \frac{\int_0^{\infty} \nu\, v(\nu)v^*(\nu)d\nu}{\int_0^{\infty} v(\nu)v^*(\nu)d\nu}, \quad (\Delta\nu)^2 = \frac{\int_0^{\infty} (\nu-\bar{\nu})^2 v(\nu)v^*(\nu)d\nu}{\int_0^{\infty} v(\nu)v^*(\nu)d\nu}. \tag{2.13}$$

The corresponding averages $\bar{t}'$ and $\Delta't$ are defined by Gabor in terms of the analytic signal:

$$\bar{t}' = \frac{\int_{-\infty}^{+\infty} t\hat{V}(t)\hat{V}^*(t)dt}{\int_{-\infty}^{+\infty} \hat{V}(t)\hat{V}^*(t)dt}, \quad (\Delta't)^2 = \frac{\int_{-\infty}^{+\infty} (t-\bar{t}')^2\, \hat{V}(t)\hat{V}^*(t)dt}{\int_{-\infty}^{+\infty} \hat{V}(t)\, \hat{V}^*(t)dt}. \tag{2.14}$$

Now according to (2.7), $\hat{V}(t)$ and the function

$$\begin{aligned} \mathcal{F}\nu - (\nu) &= 2v(\nu) \quad \text{when } \nu \geq 0, \\ &= 0 \quad \text{when } \nu < 0, \end{aligned} \tag{2.15}$$

are (complex) Fourier transforms of each other. Consequently the well-known argument of Weyl and Pauli [6] leading to the usual inequality may be applied without change and one obtains**

$$\Delta't\,\Delta\nu \geq \frac{1}{4\pi}. \tag{2.16}$$

---

*Actually Gabor uses a slightly different normalization. Our $\Delta\nu$ and Δt correspond to $1/\sqrt{2\pi}$ times Gabor's.

**If instead of ν the circular frequency $\omega = 2\pi\nu$ is used, the more usual factor 1/2 is obtained on the right hand side:

$$\Delta't\,\Delta\omega \geq 1/2.$$

This is essentially Gabor's formulation of the inequality when the signal is real.

The quantities $\bar{\nu}$ and $\Delta\nu$ represent correctly the mean and the r.m.s. width of the spectral density function of the real signal V. However, as $\bar{t}'$ and $\Delta't$ are defined not in terms of V but in terms of the associated analytic signal $\hat{V}$, the physical significance of these quantities is not clear; nor is it clear how they are related to the corresponding averages $\bar{t}$ and Δt defined in terms of V itself. Further, since $\hat{V}\hat{V}^* = V^2 + W^2$, it follows that $\bar{t}'$ and $\Delta't$ are only defined if, in addition to the appropriate moments of V^2, the corresponding moments of the square of the Hilbert transform W of V also exist. This evidently imposes some restrictions on the real signal. We shall now investigate these questions.

3. The physical significance of $\bar{t}'$ and $\Delta't$ and a condition for their existence

Let ν_o be any frequency and let us rewrite (2.7) in the form

$$\hat{V}(t) = A(t)e^{i\Psi(t)}\, e^{-2\pi i\nu_o t} \qquad (A,\Psi \text{ real}), \tag{3.1}$$

where

$$\begin{aligned} A(t)e^{i\Psi(t)} &= 2\int_0^\infty |v(\nu)|e^{i\left[\phi(\nu)-2\pi(\nu-\nu_o)t\right]}\, d\nu \\ &= \int_{-\nu_o}^{\infty} g(\mu)e^{-2\pi i\mu t}\, d\mu, \end{aligned} \tag{3.2}$$

and

$$g(\mu) = 2|v(\nu_o+\mu)|e^{i\phi(\nu_o+\mu)} = 2v(\nu_o+\mu). \tag{3.3}$$

The functions V and W, being the real and imaginary parts of $\hat{V}$, may, according to (3.1), be written in the form

$$\begin{aligned} V(t) &= A(t)\cos\left[\Psi(t) - 2\pi\nu_o t\right], \\ W(t) &= A(t)\sin\left[\Psi(t) - 2\pi\nu_o t\right]. \end{aligned} \tag{3.4}$$

Suppose that V is a high-frequency narrow-band signal, and let ν_o be a mean frequency of the signal. Then the spectral amplitudes $|v(\nu)|$ differ appreciably from zero only when ν is in a relatively small neighborhood of the mean frequency. Consequently $|g(\mu)|$ will have appreciable values only in a small neighborhood around $\mu = 0$. Hence the integral (3.2) represents a superposition of harmonic components of low frequencies so that $A(t)$ and $\Psi(t)$ will vary slowly

in comparison with the periodic term $e^{-2\pi i\nu_o t}$. Evidently A(t) represents an envelope of the real signal, and it follows from (3.4) and (2.8) that

$$(3.5a)\qquad A(t) = \sqrt{V^2(t) + W^2(t)} = \sqrt{\hat{V}(t)\hat{V}^*(t)} = |\hat{V}(t)|,$$

$$(3.5b)\qquad \Psi(t) = 2\pi\nu_o t + \tan^{-1}\frac{W(t)}{V(t)} = 2\pi\nu_o t + \tan^{-1}\left(i\,\frac{\hat{V}^* - \hat{V}}{\hat{V}^* + \hat{V}}\right).$$

Since $\hat{V}\,\hat{V}^* = A^2$, it follows that: $\bar{t}'$ and $\Delta' t$ defined by (2.14) may be interpreted as the mean epoch and the (normalized) r.m.s. width of the square of the envelope belonging to the real signal V. However, as will now be shown, whenever these quantities exist they will also be equal to the mean epoch and to the r.m.s. width of the square of the signal itself.

Let F(x) be any function of a real variable x and let $M_n(F)$ be its n-th moment,

$$(3.6)\qquad M_n(F) = \int_{-\infty}^{+\infty} x^n F(x)dx.$$

The equations (2.14) may be written in the form

$$(3.7)\qquad \bar{t}' = \frac{M_1(\hat{V}\hat{V}^*)}{M_o(\hat{V}\hat{V}^*)}, \qquad (\Delta' t)^2 = \frac{M_o(\hat{V}\hat{V}^*)M_2(\hat{V}\hat{V}^*) - M_1^2(\hat{V}\hat{V}^*)}{M_o^2(\hat{V}\hat{V}^*)}.$$

Consider now the corresponding moments of V^2. Since according to (2.6) and (2.4)

$$(3.8)\qquad V(t) = 2\mathcal{R}\int_o^\infty v(\nu)e^{-2\pi i\nu t}\,d\nu,$$

where $\mathcal{R}$ denotes the real part, $M_n(V^2)$ may formally be expressed in the form

$$M_n(V^2) = \int_{-\infty}^{+\infty} t^n V(t)\left[2\mathcal{R}\int_o^\infty v(\nu)e^{-2\pi i\nu t}d\nu\right]dt$$

$$= 2\mathcal{R}\int_o^\infty v(\nu)d\nu \int_{-\infty}^{+\infty} t^n V(t)e^{-2\pi i\nu t}dt$$

$$= 2\mathcal{R}\int_o^\infty v(\nu)d\nu \left(-\frac{1}{2\pi i}\,\frac{\partial}{\partial\nu}\right)^n \int_{-\infty}^{+\infty} V(t)e^{-2\pi i\nu t}\,dt$$

$$(3.9) \qquad = 2\mathcal{R}\left(-\frac{1}{2\pi i}\right)^n \int_0^\infty v(\nu)\left(\frac{\partial}{\partial\nu}\right)^n v^*(\nu)d\nu,$$

where (2.1) and (2.2) were used.

Next we consider the corresponding moments of W^2. Comparison of the real representations (2.6) and (2.8) shows that V may be transformed into W simply by changing the phase $\phi(\nu)$ of each Fourier component to $\phi(\nu) - \pi/2$ $(\nu \geq 0)$. It follows that if $M_n(W^2)$ exists, its value is given by (3.9), with $v(\nu) = |v(\nu)|e^{i\phi}$ replaced by $w(\nu) = |v(\nu)|e^{i(\phi-\pi/2)} = -iv(\nu)$. But (3.9) remains invariant under this transformation. Noting also that

$$(3.10) \qquad M_n(\hat{V}\,\hat{V}^*) = M_n\left[(V+iW)(V-iW)\right] = M_n(V^2) + M_n(W^2),$$

we have the following theorem:

> Let V(t) be any real function that may be expressed as a Fourier integral. If W(t) is the Hilbert transform of V(t) and $\hat{V}(t)$ the analytic signal associated with V(t) and if, moreover, for any particular value of n (non-negative integer) the moments $M_n(V^2)$ and $M_n(W^2)$ both exist, then $M_n(\hat{V}\,\hat{V}^*)$ also exists and moreover
>
> $$(3.11) \qquad M_n(\hat{V}\,\hat{V}^*) = 2M_n(V^2) = 2M_n(W^2).$$

Let us now introduce the mean epoch $\bar{t}$ and the r.m.s. width Δt of the square of the real signal V:

$$(3.12)\begin{cases} \bar{t} = \dfrac{\int_{-\infty}^{+\infty} t\,V^2(t)dt}{\int_{-\infty}^{+\infty} V^2(t)dt} = \dfrac{M_1(V^2)}{M_o(V^2)}, \\ (\Delta t)^2 = \dfrac{\int_{-\infty}^{+\infty} (t-\bar{t})^2 V^2(t)dt}{\int_{-\infty}^{+\infty} V^2(t)dt} = \dfrac{M_o(V^2)M_2(V^2) - M_1^2(V^2)}{M_o^2(V^2)}. \end{cases}$$

Assuming that the moments of V^2 and W^2 of orders $n = 0,1,2$ exist, it follows from the theorem just established that V^2 may be replaced by $\hat{V}\,\hat{V}^*$ in (3.12), and we see on comparison with (3.7) that

$$(3.13) \qquad \bar{t}' = t, \qquad \Delta' t = \Delta t.$$

Thus the mean epoch of the squared amplitude of the envelope belonging to the real signal is equal to the mean epoch of the squared signal itself, and the r.m.s. widths are also equal, and (2.16) gives

$$(3.14) \qquad \Delta t\,\Delta\nu \geq \frac{1}{4\pi}.$$

We shall not investigate here the general conditions which must be imposed on $V(t)$ in order that the required moments should exist. We only confine ourselves to the case of great practical importance when $V(t)$ is zero outside a finite interval $-T \leq t \leq T$. Then $M_0(V^2)$, $M_1(V^2)$ and $M_2(V^2)$ exist, but as shown in the appendix, the corresponding moments involving W^2 do not all exist unless

$$(3.15a) \qquad \int_{-T}^{T} V(t)dt = 0.$$

According to (2.2) this condition may also be written as

$$(3.15b) \qquad v(0) = 0.$$

For such signals the identity (3.13) and the inequality (3.14) hold*.

---

*Since this investigation was carried out Kay and Silverman[3] have obtained a more general inequality, which holds for all real signals for which Δt and $\Delta\nu$ exist (irrespective of the existence of $\Delta' t$). In our notation their formula is

$$\Delta t\,\Delta\nu \geq \frac{1}{4\pi}(1-\mu),$$

where

$$\mu = \frac{2\bar{\nu}|v(0)|^2}{\int_{-\infty}^{+\infty} |v(\nu)|^2 d\nu}.$$

They also showed that signals exist for which the product $\Delta t\,\Delta\nu$ is smaller than $1/4\pi$.

4. A definition of the coherence time of a light beam

The results of the preceding sections will now be used to give a satisfactory definition of coherence time of polychromatic radiation. We shall discuss in detail only the case of light, but similar considerations apply for other noise-like signals, such as microwaves from a radio star, the fluctuating field of an electron stream, etc.

Let us consider a simple interference experiment. A beam of light from a source O is divided into two beams of the same intensity and the beams are brought together after a path difference

$$(4.1) \qquad \ell = c\tau$$

(c = vacuum velocity of light), has been introduced between them, for example in a Michelson interferometer. For simplicity we shall consider the light disturbance to be a real scalar, e.g., a Cartesian component of the electric field vector. We assume that the field exists only for a finite time interval $-T \leq t \leq T$ which is large compared to the basic optical periods $(1/\bar{\nu})$ and to the reciprocal of the effective bandwidth.

Apart from inessential multiplying factors, the disturbance V_P at the point P of superposition is related to the disturbance V at the point O where the beam is divided by

$$(4.2) \qquad V_P(t) = \left[V_0(t-t_1) + V_0(t-t_1-\tau)\right] ,$$

where t_1 and $t_1 + \tau$ are the times in which the light travels from O to P along the two paths. The intensity I_p observed at P is the time average of $V_P^2(t)$, and we may, for convenience, take the averaging interval equal to the interval 2T during which the field is assumed to exist*:

$$(4.3) \qquad I_P = \frac{1}{2T}\int_{-T}^{T} V_P^2(t)dt.$$

---

*Under normal experimental conditions the averaging interval is large compared to $1/\bar{\nu}$ and $1/\Delta'\nu$, and provided that, as here assumed, T is much larger than either of these quantities its exact value is evidently immaterial.

Denoting the time average by angular brackets, it follows from (4.2) and (4.3) that

$$(4.4)\qquad I(P) = \langle V_0^2(t-t_1)\rangle + \langle V_0^2(t-t_1-\tau)\rangle + 2\langle V_0(t-t_1)V_0(t-t_1-\tau)\rangle .$$

Assuming the field to be stationary, we may shift the origin of time. Setting

$$(4.5)\qquad J(\tau) = \langle V_0(t+\tau)V_0(t)\rangle ,$$

the intensity at P may be expressed in the form

$$(4.6)\qquad I_P = 2\left[J(0) + J(\tau)\right].$$

$J(0) = \langle V_0^2(t)\rangle$ is the intensity of each beam and $J(\tau)$ is an interference term. Equation (4.5) defines the interference in terms of the rapidly fluctuating function $V_0(t)$, but it represents a measurable quantity. For according to (4.6) it may be obtained from the measurements of the intensities at O and P.

We now represent $V_0(t)$ as a Fourier integral of the form (2.1). It then follows from the convolution theorem and the 'reality condition' (2.3) that $J(\tau)$ may also be expressed in the form

$$(4.7)\qquad J(\tau) = \int_0^\infty j(\nu)\cos 2\pi\nu\tau\, d\nu,$$

where

$$(4.8)\qquad j(\nu) = \frac{1}{T}\,|v(\nu)|^2$$

is the average spectral energy density. According to (4.7), $J(\tau)$ and $j(\nu)$ are Fourier cosine transforms of each other*.

Since $V_0(t)$ was assumed to be zero when $|t| > T$, it follows from (4.5) that $J(\tau) = 0$ when $|\tau| > T$; moreover, since we are dealing with a pure radiation field $j(0) = 0$. Hence if quantities $\Delta\tau$ and $\Delta\nu$ are defined by**

---

*In the limit as $T \to \infty$ this result reduces to the Wiener-Khintchine theorem [cf. Wang and Uhlenbeck[7], p.326, or S.O. Rice[8] Sec. 2.1]. We do not proceed to the limit here, since we shall make use of our theorem concerning moments of squares of Hilbert transforms, established on the assumption that the real signal is zero outside a finite interval.

**Since $J(\tau)$ is an even function of τ, $\int_{-\infty}^{+\infty} \tau J^2(\tau)d\tau = 0$ so that $\bar{\tau} = 0$.

$$(4.9a)\qquad (\Delta\tau)^2 = \frac{\int_{-\infty}^{+\infty} \tau^2 J^2(\tau)\,d\tau}{\int_{-\infty}^{+\infty} J^2(\tau)\,d\tau} = \frac{\int_{0}^{\infty} \tau^2 J^2(\tau)\,d\tau}{\int_{0}^{\infty} J^2(\tau)\,d\tau},$$

$$(4.9b)\qquad (\Delta\nu)^2 = \frac{\int_{0}^{\infty} (\nu-\bar{\nu})^2 j^2(\nu)\,d\nu}{\int_{0}^{\infty} j^2(\nu)\,d\nu}, \qquad \bar{\nu} = \frac{\int_{0}^{\infty} \nu j^2(\nu)\,d\nu}{\int_{0}^{\infty} j^2(\nu)\,d\nu},$$

it follows from the results established in Section 3 that

$$(4.10)\qquad \Delta\tau\,\Delta\nu \geq \frac{1}{4\pi}.$$

The quantity $\Delta\tau$ has an interesting interpretation in connection with the well-known method of Michelson[9] for determining the energy distribution in spectral lines from measurements of the visibility of fringes. Let

$$(4.11)\qquad \hat{J}(\tau) = \int_{0}^{\infty} j(\nu)\, e^{-2\pi i\nu\tau}\,d\nu$$

be the analytic signal associated with $J(\tau)$. Then, according to (3.4) and (3.5). J may be written in the form

$$(4.12)\qquad J(\tau) = B(\tau)\cos\left[\Phi(\tau) - 2\pi\bar{\nu}\tau\right],$$

where

$$(4.13)\qquad B(\tau) = |\hat{J}(\tau)|,$$

$$\Phi(\tau) = 2\pi\bar{\nu}\tau + \tan^{-1}\left(i\,\frac{\hat{J}^*-\hat{J}}{\hat{J}^*+\hat{J}}\right),$$

and the expression (4.6) for the intensity at the point P becomes

$$(4.14)\qquad I_P = 2\left\{\hat{J}(0) + |\hat{J}(\tau)|\cos\left[\Phi(\tau) - 2\pi\bar{\nu}\tau\right]\right\}.$$

If $\Delta\nu/\bar{\nu} \ll 1$, then $|\hat{J}(\tau)|$ and $\Phi(\tau)$ will vary slowly with τ in comparison with $\cos(2\pi\bar{\nu}\tau)$ and $\sin(2\pi\bar{\nu}\tau)$ and we see that the minima and maxima of the intensity near P are approximately given by

$$
\begin{aligned}
I_{max} &= 2\left[\hat{J}(0) + |\hat{J}(\tau)|\right], \\
I_{min} &= 2\left[\hat{J}(0) - |\hat{J}(\tau)|\right].
\end{aligned}
\tag{4.15}
$$

Hence the Michelson visibility of the fringes $\mathscr{V}(\tau)$ is given by

$$
\mathscr{V}(\tau) = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \left|\frac{\hat{J}(\tau)}{\hat{J}(0)}\right| , \tag{4.16}
$$

i.e., it is proportional to the absolute value of the complex interference function $\hat{J}(\tau)$. Now according to (4.11) and the Fourier inversion formula, the energy distribution $j(\nu)$ (defined to be zero for $\nu < 0$) is the Fourier transform of $\hat{J}(\tau)$, so that as first shown by Michelson, $j(\nu)$ may be determined from measurements of the visibility curve, provided that suitable assumptions are made about the phase* $\Phi(\tau)$.

Now according to Section 3, we may replace $J^2(\tau)$ by $|\hat{J}(\tau)|^2$ in (4.9a); then using equation (4.16) it follows that

$$
(\Delta\tau)^2 = \frac{\int_0^{+\infty} \tau^2 \mathscr{V}^2(\tau)\,d\tau}{\int_0^{\infty} \mathscr{V}^2(\tau)\,d\tau} . \tag{4.17}
$$

Thus $\Delta\tau$ is equal to the normalized r.m.s. width of the square of the Michelson visibility function, and it is evidently reasonable to regard $\Delta\tau$ as the coherence time of the light. Of course, the visibility is not zero when the path difference between the beams exceeds the length $\Delta\ell = c\Delta\tau$, but is then, as a rule, small compared to the visibility at the center ($\tau = 0$) of the pattern. Other definitions of the coherence time are, naturally, possible. For example, we could define it as the τ-value beyond which $J(\tau)$ is smaller than $\varepsilon J(0)$, where ε is some prescribed number which is small compared to unity. It appears, however, that only when $\Delta\tau$ is defined as a simple average such as (4.9a) a rigorous reciprocity relation of the required type may be formulated. It is to be noted that both $\Delta\tau$ and $\Delta\nu$ can be determined from experiment: $\Delta\tau$ from the visibility curve $\mathscr{V}(\tau)$ and $\Delta\nu$ from the spectral

---

*$\Phi(\tau)$ may in principle be determined from measurements of the position of the fringes, but such measurements are extremely difficult to carry out.

energy curve $j(\nu)$.

It is often convenient to describe an interference experiment in terms of mutually incoherent 'elementary wave trains' of finite length, rather than in terms of the random function $V(t)$, or its auto-correlation function $J(\tau)$. The coherence time as here defined may evidently be interpreted as the average duration of the equivalent wave trains.

Since our definition of the coherence time involves the second power of J and therefore the fourth power of V, it may not be expected at first sight, that a physical interpretation could be given to this quantity. However, if we write $V(t) = V_1$, $V(t+\tau) = V_2$ and denote by $P(V_1,V_2)dV_1dV_2$ the probability that for any t (τ fixed), V_1 will be found in the interval V_1, V_1+dV_1 and V_2 in the interval V_2, V_2+dV_2, then under usual circumstances $P(V_1,V_2)$ may be assumed to be a Gaussian distribution and it may then be shown* that

$$J^2(\tau) = \tfrac{1}{2}\langle \Delta I(t+\tau)\,\Delta I(t)\rangle , \tag{4.18}$$

where

$$\Delta I(t) = I(t) - \langle I(t)\rangle = V^2(t) - \langle V^2(t)\rangle . \tag{4.19}$$

Thus $J^2(\tau)$ is proportional to the correlation that exists between the fluctuations in the intensity at two instants of time separated by an interval τ, and it follows that (4.9a) may be written in the form

$$(\Delta\tau)^2 = \frac{\int_0^\infty \tau^2 \langle \Delta I(t+\tau)\,\Delta I(t)\rangle \, d\tau}{\int_0^\infty \langle \Delta I(t+\tau)\,\Delta I(t)\rangle \, d\tau} , \tag{4.20}$$

Returning to the basic relation (4.10) it is not difficult to see that with the type of spectral distributions normally encountered in optics, the inequality may be replaced by the order of magnitude sign. To see this, consider the spectrum which has the form of a truncated Gaussian function

$$j(\nu) = e^{-(\nu-\nu_o)^2/2\sigma^2} \qquad (\nu \geq 0). \tag{4.21}$$

*See Wolf [10]. Blanc-Lapierre and Dumontet [11] pointed out that P may not be Gaussian for points close enough to the source, or with light of very low intensity.

Now strictly speaking this is not a function of the class that we are considering, since it does not obey the condition $j(0) = 0$. However if $\sigma/\nu_o \ll 1$, $j(0)$ will be negligible in comparison with $j(\nu_o)$ and we may assume that the corresponding value of the product $\Delta t \Delta \nu$ will be a fairly representative one.

On substituting from (4.21) into (4.7) and setting $\mu = (\nu-\nu_o)/\sigma\sqrt{2}$, we obtain for the interference function the expression

$$(4.22) \qquad J(\tau) = \sigma\sqrt{2} \int_{-\nu_o/\sigma\sqrt{2}}^{\infty} e^{-\mu^2} \cos\left[2\pi(\nu_o + \sigma\sqrt{2}\,\mu)\tau\right] d\mu.$$

Now since $\nu_o/\sigma \gg 1$, the lower limit may be replaced by $-\infty$ without introducing an appreciable error and it follows that

$$(4.23) \qquad J(\tau) \sim \sqrt{2\pi}\,\sigma e^{-2\pi^2\sigma^2\tau^2} \cos 2\pi\nu_o\tau .$$

With this approximation,

$$(4.24) \quad \begin{cases} \displaystyle\int_o^{\infty} J^2(\tau)d\tau = \frac{1}{4}\sqrt{\pi}\,\sigma\left(1 + e^{-\nu_o^2/\sigma^2}\right), \\ \displaystyle\int_o^{\infty} \tau^2 J^2(\tau)d\tau = \frac{1}{32\pi\sqrt{\pi}\,\sigma}\left\{1 + \left[1 - 2\left(\frac{\nu_o}{\sigma}\right)^2\right] e^{-\nu_o^2/\sigma^2}\right\} . \end{cases}$$

Neglecting terms which involve the exponential factor, it follows that

$$(4.25) \qquad (\Delta\tau)^2 \sim \frac{1}{8\pi^2\sigma^2} .$$

With a similar approximation,

$$(4.26) \qquad (\Delta\nu)^2 \sim \frac{\sigma^2}{2} ,$$

and it follows that

$$(4.27) \qquad \Delta\tau\Delta\nu \sim \frac{1}{4\pi} .$$

Finally it should be noted that the effective width $\Delta\nu$ is not the usual

r.m.s. width of the spectral energy function $j(\nu) = \frac{1}{T}|v(\nu)|^2$, but rather the r.m.s. width of the square of this function. As a rule the two r.m.s. widths are, however, of the same order of magnitude. For the full Gaussian function, for example, they differ by a multiplicative factor $\sqrt{2}$.

5. A generalization

In the preceding section we considered interference between two beams obtained by the division of a single beam at a particular point O. The interference effects were then characterized by the function $J(\tau)$ which depends on the coordinates of O and on the time delay τ introduced between the beams.

In the general theory of partially coherent fields it is necessary to employ a more general correlation function*

$$J_{12}(\tau) = \langle V_1(t+\tau)V_2(t)\rangle \tag{5.1}$$

which depends on the coordinates of two points O_1 and O_2 and on a time delay τ. This correlation function may be determined from a Young interference experiment with a small opening at each of these two points**. We shall now formulate a reciprocity relation appropriate to experiments in which the interference effects are characterized by $J_{12}(\tau)$.

We again assume that the field exists only for a finite time interval $-T \leq t \leq T$, and we represent the (real) disturbance at the points O_1 and O_2 as Fourier integrals

$$V_1(t) = \int_{-\infty}^{+\infty} v_1(\nu)e^{-2\pi i\nu t}d\nu, \quad V_2(t) = \int_{-\infty}^{+\infty} v_2(\nu)e^{-2\pi i\nu t}d\nu. \tag{5.2}$$

Then, using the convolution theorem and the reality conditions $v_1(-\nu) = v_1^*(\nu)$, $v_2(-\nu) = v_2^*(\nu)$, (5.1) can readily be expressed in the following 'spectral form':

$$J_{12}(\tau) = \int_0^{\infty} |j_{12}(\nu)| \cos\left[\phi_{12}(\nu) - 2\pi\nu\tau\right]d\nu, \tag{5.3}$$

where

$$j_{12}(\nu) = |j_{12}(\nu)|e^{i\phi_{12}(\nu)} = \frac{1}{T}\left[v_1(\nu)v_2^*(\nu)\right]. \tag{5.4}$$

---

*cf. Blanc-Lapierre and Dumontet[11] and Wolf[12].

**For details see Wolf[12] or Thompson and Wolf[13].

Formulas (5.3) and (5.4) are generalizations of the relations (4.7) and (4.8) and express the correlation function $J_{12}(\tau)$ as the Fourier transform of the averaged 'mutual spectral energy' $|j_{12}(\nu)|$ and the associated phase $\phi_{12}(\nu)$.

We now define the quantities

$$(5.5a)\qquad \bar{\tau}_{12} = \frac{\int_{-\infty}^{+\infty} \tau J_{12}^2(\tau)d\tau}{\int_{-\infty}^{+\infty} J_{12}^2(\tau)\,d\tau}, \quad \left(\Delta\tau_{12}\right)^2 = \frac{\int_{-\infty}^{+\infty} (\tau-\bar{\tau})^2 J_{12}^2(\tau)d\tau}{\int_{-\infty}^{+\infty} J_{12}^2(\tau)d\tau},$$

$$(5.5b)\qquad \bar{\nu}_{12} = \frac{\int_{0}^{\infty} \nu\, j_{12}(\nu)\, j_{12}^*(\nu)d\nu}{\int_{0}^{\infty} j_{12}(\nu)\, j_{12}^*(\nu)d\nu}, \quad \left(\Delta\nu_{12}\right)^2 = \frac{\int_{0}^{\infty} (\nu-\bar{\nu})^2\, j_{12}(\nu)\, j_{12}^*(\nu)d\nu}{\int_{0}^{\infty} j_{12}(\nu)\, j_{12}^*(\nu)d\nu}.$$

Since according to (5.1), $J_{12} = 0$ when $|\tau| > T$, and moreover, since $j_{12}(0)$ may again be assumed to be zero, we have, in accordance with Section 3

$$(5.6)\qquad \Delta\tau_{12}\Delta\nu_{12} \geq \frac{1}{4\pi}.$$

It also follows from Section 3 that in (5.5a) we may replace J_{12}^2 by $\hat{J}_{12}\hat{J}_{12}^*$, where $\hat{J}_{12}$ is the analytic signal associated with J_{12}, i.e.*

$$(5.7)\qquad \hat{J}_{12}(\tau) = \int_{0}^{\infty} |j_{12}(\nu)| e^{i\left[\phi_{12}(\nu) - 2\pi\nu\tau\right]} d\nu.$$

In a similar manner as in Section 4, (4.16), it is possible to relate $|J_{12}(\tau)|$ to the visibility of fringes. If the intensity of the beams proceeding from the openings at O_1 and O_2 are equal and if $\mathcal{V}_{12}(\tau)$ represents the visibility of the fringes in the neighborhood of a point P such that $\overline{O_2P} - \overline{O_1P} = c\tau$, then

---

*It was shown elsewhere (ref. [12], p. 262), that in terms of the analytic signals $\hat{V}$ of V,

$$\hat{J}_{12}(\tau) = \tfrac{1}{2}\left\langle \hat{V}_1(t+\tau)\, \hat{V}_2^*(t)\right\rangle.$$

$$(5.8) \qquad \mathcal{V}_{12}(\tau) = \left|\frac{\hat{J}_{12}(\tau)}{\hat{J}_{12}(0)}\right|.$$

It follows that (5.5a) may also be written in the form

$$(5.9) \qquad \bar{\tau}_{12} = \frac{\int_{-\infty}^{+\infty} \tau\, \mathcal{V}_{12}^2(\tau)d\tau}{\int_{-\infty}^{+\infty} \mathcal{V}_{12}^2(\tau)d\tau}, \quad (\Delta\tau_{12})^2 = \frac{\int_{-\infty}^{+\infty} (\tau - \bar{\tau}_{12})^2 \mathcal{V}_{12}^2(\tau)d\tau}{\int_{-\infty}^{+\infty} \mathcal{V}_{12}^2(\tau)d\tau}.$$

Further, by analogy with (4.18), $\Delta\tau_{12}$ may be expressed in terms of the intensity fluctuations

$$(5.10) \qquad \begin{cases} \Delta I_1(t) = I_1(t) - \langle I_1(t)\rangle = V_1^2(t) - \langle V_1^2(t)\rangle, \\ \Delta I_2(t) = I_2(t) - \langle I_2(t)\rangle = V_2^2(t) - \langle V_2^2(t)\rangle, \end{cases}$$

at the points O_1 and O_2. For, under usual circumstances, the joint probability distribution $P(V_1,V_2)$ is Gaussian and it may then be shown that

$$(5.11) \qquad J_{12}^2(\tau) = \tfrac{1}{2}\langle \Delta I_1(t+\tau)\Delta I_2(t)\rangle;$$

equation (4.18) is a special case of this formula. Thus in (5.5a) $J_{12}^2(\tau)$ may be replaced by the function $\langle \Delta I_1(t+\tau)\Delta I_2(t)\rangle$ which expresses the correlation between the intensity fluctuations at O_1 and O_2.

Evidently $\Delta\tau_{12}$ may be interpreted as the 'mutual coherence time' of the light vibrations at O_1 and O_2, and $\Delta\nu_{12}$ as their 'effective mutual spectral width.'

Appendix

Conditions for the existence of certain moments involving Hilbert transforms

Let $V(t)$ be a real function which is zero for all values of t outside the range $-T \leq t \leq T$, and let $W(t)$ be its Hilbert transform:

$$W(t) = \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{V(t')}{t'-t}\, dt', \tag{A1}$$

where P denotes the Cauchy principle value at $t' = t$. We shall examine the existence of certain moments involving W.

We have, for all values of $|t|$ greater than T,

$$W(t) = \frac{1}{\pi} \int_{-T}^{T} \frac{V(t)}{t'-t}\, dt' = -\frac{1}{\pi t} \int_{-T}^{T} \frac{V(t')}{\left(1 - \frac{t'}{t}\right)} dt'$$

$$= -\frac{1}{\pi t} \int_{-T}^{T} V(t') \sum_{n=0}^{\infty} \left(\frac{t'}{t}\right)^n dt = -\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{M_n(V)}{t^{n+1}}\, dt, \tag{A2}$$

where

$$M_n(V) = \int_{-T}^{T} t^n V(t')dt' \tag{A3}$$

is the n-th moment of V. Hence for large values of $|t|$,

$$W(t) \sim -\frac{1}{\pi} \frac{M_{n_o}(V)}{t^{n_o+1}}, \tag{A4}$$

where n_o is the lowest integer for which $M_{n_o}(V) \neq 0$.

It follows that the integral

$$M_n(W^2) = \int_{-\infty}^{+\infty} t^n W^2(t)dt \tag{A5}$$

is convergent only if

$$n < 2n_o + 1. \tag{A6}$$

We have thus the following theorem:

If $V(t)$ is zero outside the range $-T \leq t \leq T$, and $W(t)$ is the Hilbert transform of V, then the n-th moment of W^2 will exist only if n is smaller than $2n_o + 1$, where n_o is the lowest integer for which $M_{n_o}(V) \neq 0$. In particular the moments of W^2 up to and including the second order ($n \leq 2$) will exist only if $n_o > 0$, i.e., if

$$(A7) \qquad M_o(V) = \int_{-T}^{T} V(t)dt = 0.$$

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